On determinism and well-posedness in multiple time dimensions

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Abstract: We study the initial value problem for the wave equation and the ultrahyperbolic equation for data posed on an initial hypersurface surface of mixed space- and timelike signature. We show that under a nonlocal constraint, the initial value problem posed on codimension-one hypersurfaces — the Cauchy problem — has global unique solutions in the Sobolev spaces H^m. Thus it is well-posed. In contrast, we show that the initial value problem on higher codimension hypersurfaces is ill-posed due to failure of uniqueness, at least when specifying a finite number of derivatives of the data. This failure is in contrast to a uniqueness result for data given in an arbitrary neighborhood of such initial hypersurfaces, which Courant deduces from Asgeirsson's mean value theorem. We give a generalization of Courant's theorem which extends to a broader class of equations. The proofs use Fourier synthesis and the Holmgren–John uniqueness theorem.

Keywords: ultrahyperbolic equation, nonlocal constraint

1. Introduction

The field equation

$$\triangle u - \partial_y^2 u = 0$$

in Minkowski spacetime is of central physical importance, as it describes the propagation of many of the physical quantities described by field theories, including the components of the electromagnetic field in a vacuum. Its generalization to a theory which has multiple times is an ultrahyperbolic equation. The study of these equations provides a useful window onto the mathematical status of physical theories involving multiple times, and perhaps more importantly, provides insight into the extent to which the ordinary concepts of causality and determinism survive the transition to multiple time dimensions.

Consideration of theories with multiple times has been relatively rare because it is widely believed that they are inherently unstable, and thus are not deterministic in a physically meaningful sense. Certain significant developments in theoretical physics, notably string theory, require additional dimensions, and in most work to

date† the signature for the extra dimensions is spatial, reflecting in part this concern with instability. Motivated by this, the purpose of the present paper is to reconsider the questions of stability, uniqueness and determinism of the initial value problem in the presence of multiple time dimensions. We take the model field equation in this setting to be the simple generalizations of the wave equation to multiple times, the ultrahyperbolic equation. We find that the issue of stability and uniqueness for the Cauchy problem can be addressed by imposing nonlocal constraints that arise naturally from the field equations.

It may be thought reasonable to go beyond the traditional Cauchy problem, and give initial data on hypersurfaces of higher codimension. We show that under the above constraints one can preserve stability in this setting, but uniqueness is lost, and thus determinism. Indeed, one may specify an arbitrary finite number of normal derivatives of the solution on the higher codimension hypersurface, and insist upon smooth solutions, yet still fail to achieve uniqueness. In contrast to this, we conclude with a result that essentially recovers and generalizes a theorem of Courant, which shows that the values of a solution in an arbitrarily small neighborhood of the initial hypersurface are sufficient to determine the solution uniquely. In related and prior work, Woodhouse (1992) studied the case of two space and two time dimensions with initial data on a spacelike hypersurface (thus of codimension 2), using the Penrose twistor transform in the real case. He also recovered the uniqueness result of Courant and its implicit constraints on well-posed inital data for the Cauchy problem. Our work provides a rigorous analytic alternative for his solution method, which is not restricted to this choice of space and time dimensions. We remark that none of these results rely upon properties of analyticity of the data or the solution.

To fix our notation, the wave equation in d_1-many space dimensions and one time dimension is

$$\triangle_x u - \partial_y^2 u := \sum_{j=1}^{d_1} \partial_{x_j}^2 u - \partial_y^2 u = 0 \ . \tag{1.1}$$

The standard Cauchy problem is posed on $N = \{(x, y) \in \mathbb{R}_x^{d_1} \times \mathbb{R}_y^1 : y = 0\}$, a spacelike codimension one linear hypersurface, for initial data

$$u(x, 0) = f(x),\ \partial_y u(x, 0) = g(x).$$

A nonstandard Cauchy problem is posed for a linear hypersurface of mixed signature $N = \{(x, y) : x_1 = 0\} \subseteq \mathbb{R}_x^{d_1} \times \mathbb{R}_y^1$, namely

$$u(0, x', y) = f(x', y),\ \partial_{x_1} u(0, x', y) = g(x', y),$$

where the notation is that $x = (x_1, x') \in \mathbb{R}^{d_1}$. Courant (1962) calls this the non-spacelike Cauchy problem, but to avoid confusion with the non-characteristic Cauchy problem, we call it a Cauchy problem of *mixed signature*.

An ultrahyperbolic equation has the form

$$\triangle_x u - \triangle_y u := \sum_{j=1}^{d_1} \partial_{x_j}^2 u - \sum_{j=1}^{d_2} \partial_{y_j}^2 u = 0 \ , \tag{1.2}$$

† Exceptions include the work of Tegmark (1997), Hull (1999), Hull & Khuri (2000), and Bars (2001).

where $x \in \mathbb{R}^{d_1}$ are considered to be the spacelike variables and $y \in \mathbb{R}^{d_2}$ are time-like. The Cauchy problem considers initial data posed on a linear hypersurface of codimension one. Choosing y_1 as the direction of evolution, Cauchy data consist of

$$u(x,0,y') = f(x,y'),\ \partial_{y_1} u(x,0,y') = g(x,y')$$

on the hypersurface $N = \{(x,y) \in \mathbb{R}^{d_1}_x \times \mathbb{R}^{d_2}_y : y_1 = 0\}$.

The initial value problem on a *higher* codimension hypersurface M could take various forms. A natural problem from the perspective of theories with multiple times is to consider the spacelike hypersurface $M = \{(x,y) \in \mathbb{R}^{d_1}_x \times \mathbb{R}^{d_2}_y : y = 0\}$ of codimension d_2. Alternatively, one may consider more general $M = \{(x,y) \in \mathbb{R}^{d_1}_x \times \mathbb{R}^{d_2}_y : x_{p_1+1} = \cdots = x_{d_1} = 0,\ y_{P_2+1} = \cdots = y_{d_2-1} = 0\}$ where $0 \le p_1 \le d_1$ and $0 \le p_2 \le d_2 - 1$. There is in either case a question as to how much data, and what constraints, are to be considered on M. Some of the options are to (i) give the zeroth and first normal derivatives of u on M, (ii) give some finite number of derivatives of u on M which are compatible with the constraint imposed by the ultrahyperbolic equation, or (iii) specify infinitely many derivatives of u on M. In this paper we consider the first two of these cases.

An outline of the results of this paper is as follows. In section 2 we use Fourier methods to show that the Cauchy problem for the ultrahyperbolic equation (1.2) is ill-posed in general but well-posed on Sobolev spaces H^m if an explicit nonlocal constraint is imposed upon the Cauchy data. This applies as well to the wave equation with Cauchy data on a mixed signature hypersurface. In section 3 we consider the initial value problem for data given on higher codimension hypersurfaces, and we find that solutions are highly nonunique for the initial value problems of type (i) and (ii) above, even among H^m smooth solutions and with the imposition of the constraint given in section 2. In particular, for theories with multiple times that can be transformed to the form of equation (1.2), data posed on the hypersurface $M = \{y = 0\}$ do not uniquely determine the solution at any other point in time $y \in \mathbb{R}^{d_2} \backslash \{0\}$. The extension problem for higher numbers of derivatives is treated by the same method as case (i) of zeroth and first normal derivatives. Regarding case (iii), in which one specifies infinitely many derivatives on the initial hypersurface M, we do not have an answer. We do show in section 4 that among smooth solutions, data in an arbitrarily small ellipsoidal neighborhood of a disk in M uniquely determine the data in the envelope of its light cones. This is analogous to a result in Courant (1962) that is derived from Asgeirsson's mean value theorem.

2. The Cauchy problem

Let $x \in \mathbb{R}^{d_1}$ and $y \in \mathbb{R}^{d_2}$ be the Cartesian coordinates of space-time, denote $y = (y_1, y')$ and consider the Cauchy problem of evolution in the coordinate y_1. The Cauchy problem of mixed signature that we address is posed as

$$\partial^2_{y_1} u = \triangle_x u - \triangle_{y'} u\ , \tag{2.1}$$

with Cauchy data $u(x,0,y') = u_0(x,y')$ and $\partial_{y_1}\ u(x,0,y') = u_1(x,y')$. The standard Sobolev spaces H^m of functions of the variables (x,y') are defined as closures of

$C_0^\infty(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2-1})$ with respect to the norms

$$\|f\|_m^2 = \sum_{|\alpha|+|\beta|\le m} \int \left|\partial_x^\alpha \partial_{y'}^\beta f(x,y')\right|^2 dxdy' \ .$$

Additionally, there is an energy functional, of indefinite sign, that is associated with equation (2.1), namely

$$E(u) := \frac{1}{2}\iint |\partial_{y_1} u|^2 + |\nabla_x u|^2 - |\nabla_{y'} u|^2 \, dxdy' \ .$$

Theorem 2.1. *Suppose that the evolution mapping* $y_1 \to \binom{u(x,y_1,y')}{\partial_{y_1} u(x,y_1,y')}$ *is in* $C^1(\mathbb{R}_{y_1} : H^1 \times H^0)$*. Then the energy is conserved along a solution* $u(\cdot, y_1, \cdot)$*:*

$$E(u(\cdot, y_1, \cdot)) = E(u(\cdot, 0, \cdot)).$$

Proof. Given $\binom{u(x,y_1,y')}{\partial_{y_1} u(x,y_1,y')} \in C^1$, the following calculation is justified:

$$\begin{aligned} \partial_{y_1} E(u) \; &= \iint (\partial_{y_1} u \cdot \partial_{y_1}^2 u + \nabla_x u \cdot \nabla_x \partial_{y_1} u - \nabla_{y'} u \cdot \nabla_{y'} \partial_{y_1} u) dxdy' \\ &= \iint \partial_{y_1} u (\partial_{y_1}^2 u + \triangle_x u + \triangle_{y'} u) dxdy' \\ &= 0 \ . \end{aligned}$$

□

The key issue is that the Cauchy problem above for equation (2.1) is ill-posed for $d_2 \ge 2$ and solutions are *not* in general in $C^1(\mathbb{R}_{y_1} : H^1 \times H^0)$. The energy is indefinite and in particular not bounded below, hence it does not in general define an energy norm with which to control the Sobolev norms of solutions of the evolution equations.

To move to the next level of analysis, we give a Fourier synthesis of the evolution operator for the Cauchy problem of mixed signature. Given $\binom{u_0}{u_1} \in H^{m+1} \times H^m$, consider the Fourier space variables $(x, y') \to (\xi, \eta')$ and define the Fourier transform in the standard way,

$$\begin{pmatrix} \hat{u}_0(\xi,\eta') \\ \hat{u}_1(\xi,\eta') \end{pmatrix} = \frac{1}{\sqrt{2\pi}^d} \iint e^{-i\xi\cdot x} e^{-i\eta'\cdot y'} \begin{pmatrix} u_0(x,y') \\ u_1(x,y') \end{pmatrix} dxdy'$$

where $d = (d_1 + d_2 - 1)$. On a formal level equation (2.1) under Fourier transform will read

$$\partial_{y_1}^2 \hat{u} = (-|\xi|^2 + |\eta'|^2)\hat{u} \ ,$$

giving rise to the expression for the propagator, $\exp(y_1\sqrt{\triangle_x - \triangle_{y'}})$. The solution thus reads

$$\hat{u}(\xi, y_1, \eta') = \cos(\sqrt{|\xi|^2 - |\eta'|^2} y_1)\hat{u}_0(\xi,\eta') + \frac{\sin(\sqrt{|\xi|^2 - |\eta'|^2} y_1)}{\sqrt{|\xi|^2 - |\eta'|^2}} \hat{u}_1(\xi,\eta')$$

for $|\eta'| \leq |\xi|$, while

$$\hat{u}(\xi, y_1, \eta') = \cosh(\sqrt{|\eta'|^2 - |\xi|^2} y_1)\hat{u}_0(\xi, \eta') + \frac{\sinh(\sqrt{|\eta'|^2 - |\xi|^2} y_1)}{\sqrt{|\eta'|^2 - |\xi|^2}}\hat{u}_1(\xi, \eta')$$

for $|\xi| < |\eta'|$. That is, the dispersion relation

$$\omega(\xi, \eta') = \sqrt{|\xi|^2 - |\eta'|^2} \tag{2.2}$$

holds in the Fourier space region $\{|\eta'| \leq |\xi|\}$, while in the complementary region the evolution of a Fourier mode is described by the Lyapunov exponent

$$\lambda(\xi, \eta') = \sqrt{|\eta'|^2 - |\xi|^2}. \tag{2.3}$$

When the propagator is applied to data $\binom{u_0}{u_1}$ which is analytic, this solution exists for at least short time; for analytic data of exponential type, the solution is global. However, it is clear that general initial data in $H^{m+1} \times H^m$ do not even give rise to solutions which are tempered distributions for any nonzero y_1.

On the other hand, imposing a constraint on the initial data, the solution process is well defined. The fact that some constraint is necessary is indeed evident from the Asgeirsson mean value theorem, and its consequences, as discussed in Courant (1962). The form of this nonlocal constraint is evident from the Fourier synthesis, as we shall now see.

Define a phase space X using an energy norm adapted to the propagator of equation (2.1). Using the definition of the dispersion relation (2.2) and the Lyapunov exponent (2.3), and the Plancherel identity, set $v = \binom{v_0}{v_1}$ and

$$\begin{aligned}\|v\|_X^2 := &\iint\limits_{\{|\eta'|<|\xi|\}} \omega^2(\xi, \eta')\, |\hat{v}_0(\xi, \eta')|^2 \, d\xi d\eta' \\ &+ \iint\limits_{\{|\xi|\leq|\eta'|\}} \lambda^2(\xi, \eta')\, |\hat{v}_0(\xi, \eta')|^2 \, d\xi d\eta' \\ &+ \iint |\hat{v}_1(\xi, \eta')|^2 \, d\xi d\eta' \ .\end{aligned}$$

This is a norm, unlike the actual energy associated with the equation (2.1), and can be used to control solutions when the propagator is restricted to the appropriate stable and/or unstable subspaces of X. Define

$$X^S = \left\{ v = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \in X : \ \frac{1}{2}\left(\hat{v}_0(\xi, \eta') + \frac{\hat{v}_1(\xi, \eta')}{\lambda(\xi, \eta')}\right) = 0 \text{ for } \ |\xi| < |\eta'| \right\} \tag{2.4}$$

$$X^U = \left\{ v \in X : \ \frac{1}{2}\left(\hat{v}_0(\xi, \eta') - \frac{\hat{v}_1(\xi, \eta')}{\lambda(\xi, \eta')}\right) = 0 \text{ for } \ |\xi| < |\eta'| \right\} \tag{2.5}$$

and

$$\begin{aligned}X^C &= \left\{ v \in X : \operatorname{supp}\begin{pmatrix} \hat{v}_0 \\ \hat{v}_1 \end{pmatrix}(\xi, \eta') \subseteq \ \ \{|\xi| > |\eta'|\} \right\} \\ &= X^S \cap X^U.\end{aligned}$$

The subspace X^S corresponds to the center stable subspace for evolution in $y_1 \in \mathbb{R}^+$, the subspace X^U corresponds to the center unstable subspace, and X^C is the center subspace. This nomenclature is supported by the following theorem.

Theorem 2.2. *For $\binom{u_0}{u_1} \in X^S$, the Cauchy problem of mixed signature for equation (2.1) has a unique solution in X for all $y_1 \in \mathbb{R}^+$. For $\binom{u_0}{u_1} \in X^U$ the problem has a unique solution for all $y_1 \in \mathbb{R}^-$, and whenever $\binom{u_0}{u_1} \in X^C$ the solution exists globally in $y_1 \in \mathbb{R}$. In each of these cases, the map $y_1 \to u(x, y_1, y')$ is C^1.*

Denote the propagators by Φ^S, Φ^U and Φ^C for data in the respective subspaces. These solutions are continuous with respect to their Cauchy data taken in the respective subspaces. This is the result of the next theorem.

Theorem 2.3. *Given two phase space points $u = \binom{u_0}{u_1}, v = \binom{v_0}{v_1} \in X^S$, then for $y_1 \geq 0$,*

$$\left\|\Phi^S_{y_1}(u) - \Phi^S_{y_1}(v)\right\|^2_X \leq \|u - v\|^2_X \ . \tag{2.6}$$

The analogous estimate holds for $u, v \in X^U$, for $y_1 \leq 0$:

$$\left\|\Phi^U_{y_1}(u) - \Phi^U_{y_1}(v)\right\|^2_X \leq \|u - v\|^2_X \ . \tag{2.7}$$

For $u \in X^C$, $\Phi^C_{y_1} = \Phi^S_{y_1}$ for $y_1 \geq 0$ and $\Phi^C_{y_1} = \Phi^U_{y_1}$ for $y_1 \leq 0$, and equality holds in both (2.6) and (2.7).

Proof. It suffices in Theorem 2.3 to prove the first statement. In X^S the solution has two components, distinguished by their Fourier support. Consider first $\binom{u_0}{u_1}$ such that $\mathrm{supp}(\hat{u}_0, \hat{u}_1) \subseteq \{|\xi| \geq |\eta'|\} := R_1$, which gives the center component of the evolution. The propagator is expressed

$$\mathcal{F}\Phi^S_{y_1}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} \cos(\omega y_1) & \frac{\sin(\omega y_1)}{\omega} \\ -\omega \sin(\omega y_1) & \cos(\omega y_1) \end{pmatrix} \begin{pmatrix} \hat{u}_0 \\ \hat{u}_1 \end{pmatrix}$$

where $\omega = \omega(\xi, \eta')$ is the dispersion relation (2.2). Evaluating this in the energy norm,

$$\begin{aligned} \left\|\Phi^S_{y_1}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}\right\|^2_X &= \iint \left|\cos(\omega y_1)\hat{u}_0 + \frac{\sin(\omega y_1)}{\omega}\hat{u}_1\right|^2 \omega^2 \\ &\quad + \left|-\omega \sin(\omega y_1)\hat{u}_0 + \cos(\omega y_1)\hat{u}_1\right|^2 d\xi d\eta' \\ &= \iint (|\hat{u}_0|^2 \omega^2 + |\hat{u}_1|^2) d\xi d\eta' \\ &= \left\|\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}\right\|^2_X \ . \end{aligned} \tag{2.8}$$

The propagator on the complementary space is more sensitive. Let us suppose that $\mathrm{supp}(\hat{u}_0, \hat{u}_1) \subseteq \{|\eta'| > |\xi|\}$, then $\lambda(\xi, \eta') > 0$ and we express the propagator in terms of its Fourier transform as

$$\begin{aligned} \mathcal{F}\Phi^S_{y_1}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} &= \begin{pmatrix} \cosh(\lambda y_1) & \frac{\sinh(\lambda y_1)}{\lambda} \\ \lambda \sinh(\lambda y_1) & \cosh(\lambda y_1) \end{pmatrix} \begin{pmatrix} \hat{u}_0 \\ \hat{u}_1 \end{pmatrix} \\ &= \frac{e^{\lambda y_1}}{2} \begin{pmatrix} 1 & \frac{1}{\lambda} \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} \hat{u}_0 \\ \hat{u}_1 \end{pmatrix} + \frac{e^{-\lambda y_1}}{2} \begin{pmatrix} 1 & -\frac{1}{\lambda} \\ -\lambda & 1 \end{pmatrix} \begin{pmatrix} \hat{u}_0 \\ \hat{u}_1 \end{pmatrix} . \end{aligned}$$

The subspace X^S consists of precisely those data which lie in the null space of the first term; this is the expression of the constraint

$$\lambda \hat{u}_0(\xi, \eta') + \hat{u}_1(\xi, \eta') = 0 \tag{2.9}$$

Measuring the remaining term in energy norm, we find

$$\begin{aligned} \left\| \Phi^S_{y_1} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \right\|^2_X &= \iint \frac{e^{-2\lambda y_1}}{4} \left[\left| \hat{u}_0 - \frac{\hat{u}_1}{\lambda} \right|^2 \lambda^2 + \left| -\lambda \hat{u}_0 + \hat{u}_1 \right|^2 \right] d\xi d\eta' \\ &\le \iint e^{-2\lambda y_1} \left(|\hat{u}_0|^2 \lambda^2 + |\hat{u}_1|^2 \right) d\xi d\eta' \ . \end{aligned}$$

Since we consider the propagator $\Phi^S_{y_1}$ for $y_1 \ge 0$, the exponent $-\lambda y_1$ is negative, and therefore

$$\left\| \Phi^S_{y_1} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \right\|^2_X \le \left\| \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \right\|^2_X$$

for $u = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in X^S$. For general data in X^S, one decomposes it into its components with support in $\{|\xi| \ge |\eta'|\}$ for which we use (2.8), and its component supported in $\{|\eta'| > |\xi|\}$, which in addition satisfies the constraint (2.9). Therefore on X^S

$$\left\| \Phi^S_{y_1}(u) \right\|^2_X \le \|u\|^2_X \ .$$

Bounded operators on X^S are continuous with respect to $u \in X^S$, and it is easy to see that the solution behaves continuously with respect to $y_1 \ge 0$ as well. The case for the subspace X^U is proved by the same arguments, after reversing time $y_1 \longrightarrow -y_1$. This proves Theorems 2.2 and 2.3. We remark that on the center subspace X^C, which yields global solutions, both constraints are imposed

$$\lambda \hat{u}_0(\xi, \eta') \pm \hat{u}_1(\xi, \eta') = 0 \ , \tag{2.10}$$

implying that $\hat{u}_0(\xi, \eta') = 0 = \hat{u}_1(\xi, \eta')$ for all $\{|\xi| \le |\eta'|\}$. □

The proof extends to the initial value problem posed in higher energy spaces, defined by

$$\|v\|^2_{X^m} := \sum_{|\alpha| + |\beta| \le m} \left\| \partial^\alpha_x \partial^\beta_{y'} v \right\|^2_X \ .$$

We then have

Corollary 2.4. *The higher energy space X^m decomposes into three subspaces, $X^{m,S}$, $X^{m,U}$ and $X^{m,C} = X^{m,S} \cap X^{m,U}$ such that for $u, v \in X^{m,S}$ and $y_1 \ge 0$*

$$\left\| \Phi^S_{y_1}(u) - \Phi^S_{y_1}(v) \right\|^2_{X^m} \le \|u - v\|^2_{X^m} \ ,$$

while for $y_1 \le 0$ and $u, v \in X^{m,U}$,

$$\left\| \Phi^U_{y_1}(u) - \Phi^U_{y_1}(v) \right\|^2_{X^m} \le \|u - v\|^2_{X^m} \ .$$

For $u, v \in X^{m,C}$ both estimates hold, and a global solution exists which has properties of higher Sobolev regularity. When $m > ((d_1 + (d_2 - 1))/2) + 2$ then such solutions are known to be classical C^2 solutions by the Sobolev embedding theorem.

It is natural to estimate solutions with respect to the energy norm; indeed, it *is* the energy when restricted to the center subspace X^C. Thus the problem is well-posed in the following sense: data in X^S continuously propagates to all $y_1 \in \mathbb{R}^+$, data in X^U continuously propagates to all times $y_1 \in \mathbb{R}^-$, and data in X^C, which constitute an infinite-dimensional Hilbert space, are defined globally in time. In the case of the ordinary wave equation ($d_1 = 1$), solutions in X^C correspond to the full energy space $H^1 \times L^2$.

3. The initial value problem in higher codimension

In the presence of multiple time dimensions, spacelike hypersurfaces are necessarily of higher codimension. Therefore one might consider the initial value problem with data posed on a hypersurface of codimension greater than or equal to two. Such problems are generally ill-posed. Indeed, we show that solutions can be singular for standard classes of data. Moreover, even imposing the constraint discussed in section 2, which is the requirement of global existence, smooth solutions are highly non-unique. The purpose of this section is to study the extension problem of data posed on a non-degenerate higher codimension hypersurface M to Cauchy data on a codimension one hypersurface N. There is a lot of freedom in choosing this extension, even under the constraint equation (2.9) on the resulting Cauchy data. Other extensions can be chosen to fail to satisfy the constraint. Thus the initial value problem fails to be well-posed in several ways: resulting solutions may be singular, or they may be selected to satisfy the constraint and be regular for all $y_1 \in \mathbb{R}$, however they will not be unique.

(*a*) *Codimension 2 to codimension 1 in* $\mathbb{R}^3$

Our analysis is illustrated in the example case of $M = \{y_1 = y_2 = 0\}$ and $N = \{y_1 = 0\}$ subspaces of $\mathbb{R}^3$. We suppose that initial data for a solution $u(x, y)$ is given on M in the form

$$w(x_1) = (w_0(x_1), w_{10}(x_1), w_{01}(x_1))$$

where $w_0(x_1) = u(x_1, 0)$, $w_{10}(x_1) = \partial_{y_1} u(x_1, 0)$ and $w_{01}(x_1) = \partial_{y_2} u(x_1, 0)$, corresponding to the values of the solution and its normal derivatives on M. The object is to extend $w(x_1)$ to Cauchy data $(u_0(x_1, y_2), u_1(x_1, y_2))$ on N which satisfies

$$u_0(x_1, 0) = w_0(x_1)$$
$$u_1(x_1, 0) = w_{10}(x_1)$$

and the compatibility condition

$$\partial_{y_2} u_0(x_1, 0) = w_{01}(x_1).$$

We give extensions which satisfy the constraint (2.9), therefore giving rise to global solutions in $y_1 \in \mathbb{R}$. Such extensions are nonunique. Additionally, there are extensions which fail to satisfy the constraint, lying in $X^S \backslash X^C$ or $X^U \backslash X^C$ or neither.

Definition 3.1. *The extension operator is given by*

$$E(w)(x_1, y_2) := \frac{1}{2\pi} \iint e^{i(\xi x_1 + \eta' y_2)} \hat{w}(\xi) \chi(\xi, \eta') d\eta' d\xi$$

where the kernel function $\chi(\xi, \eta')$ *is chosen such that for all* ξ*,*

$$\frac{1}{\sqrt{2\pi}} \int \chi(\xi, \eta') d\eta' = 1.$$

In order to satisfy the constraint, we ask additionally that $\operatorname{supp}(\chi(\xi, \eta')) \subseteq \{|\eta'| < |\xi|\}$*. A reasonable choice is to take*

$$\chi(\xi, \eta') = \psi(\eta' / |\xi|) \frac{1}{|\xi|},$$

for $\psi(\theta) \in C_0^\infty([-1, 1])$*,* $\psi(\theta) \geq 0$ *even, and*

$$\frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \psi(\theta) d\theta = 1. \tag{3.1}$$

Theorem 3.2. *The extension operator* E *is a bounded operator on the following space of functions:*

$$\begin{aligned} E : \dot{H}^{-1/2}(M) &\to L^2(N) \\ (\dot{H}^{-1/2} \cap H^{m-1/2})(M) &\to H^m(N) \ . \end{aligned}$$

In addition, when $w \in \dot{H}^{-3/2}(M)$ *then* $y_2 E(w) \in L^2(N)$ *and furthermore*

$$y_2 E : \dot{H}^{m-3/2}(M) \to H^m(N).$$

Using the extension operator, we generate constraint-satisfying Cauchy data on N from initial data on M as follows:

$$\begin{aligned} u_0(x_1, y_2) &:= E(w_0)(x_1, y_2) + y_2 E(w_{01})(x_1, y_2) \\ u_1(x_1, y_2) &:= E(w_{10})(x_1, y_2). \end{aligned}$$

Checking that this is a legitimate choice, we have

$$\begin{aligned} u_1(x_1, 0) &= \frac{1}{2\pi} \iint e^{i\xi x_1} \hat{w}_{10}(\xi) \chi(\xi, \eta') d\xi d\eta' \\ &= \frac{1}{\sqrt{2\pi}} \int e^{i\xi x_1} \hat{w}_{10}(\xi) \left[\frac{1}{\sqrt{2\pi}} \int \psi(\eta' / |\xi|) \frac{1}{|\xi|} d\eta' \right] d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int e^{i\xi x_1} \hat{w}_{10}(\xi) \left[\frac{1}{\sqrt{2\pi}} \int \psi(\theta) d\theta \right] d\xi \\ &= w_{10}(x_1) \end{aligned}$$

because of the normalization (eqn 3.1) of ψ. Similarly,

$$u_0(x_1, 0) = E(w_0)(x_1, 0) = w_0(x_1).$$

The compatibility condition is satisfied, since

$$\begin{aligned} \partial_{y_2} u_0(x_1, 0) &= \partial_{y_2} E(w_0)(x_1, 0) + E(w_0)(x_1, 0) \\ &= \partial_{y_2} E(w_0)(x_1, 0) + w_{01}(x_1). \end{aligned}$$

The first term of the RHS vanishes because

$$\begin{aligned}\partial_{y_2} E(w_0)(x_1,0) &= \frac{1}{2\pi}\iint e^{i\xi x_1} i\eta' \hat{w}_0(\xi)\chi(\xi,\eta')d\xi d\eta' \\ &= \frac{1}{\sqrt{2\pi}}\int ie^{i\xi x_1}\hat{w}_0(\xi)\left[\frac{1}{\sqrt{2\pi}}\int \eta'\psi(\eta'/\left|\xi\right|)\frac{1}{|\xi|}d\eta'\right]d\xi \\ &= 0,\end{aligned}$$

where we have used that $\int \theta\psi(\theta)d\theta = 0$ because $\psi(\cdot)$ has been chosen to be even.

The pair of functions $(u_0(x_1,y_2), u_1(x_1,y_2))$ gives Cauchy data for the codimension one problem that is discussed in Section 2. Because of the properties of the extension, it satisfies the constraint conditions of X^C for solutions which are globally defined in y_1. In order to apply the existence theorem, the energy norm of this Cauchy data must be finite.

Theorem 3.3. *Suppose that $w_0 \in \dot{H}^{1/2}(M)$, $w_{01} \in \dot{H}^{-1/2}$ and $w_{10} \in \dot{H}^{-1/2}$. Then the energy norm of the extension $u_0 = E(w_0)(x_1,y_2) + y_2E(w_{01})(x_1,y_2)$, $u_1(x_1,y_2) = E(w_{10})(x_1,y_2)$ is finite:*

$$\|(u_0,u_1)\|^2_{X^C} \le C(\|w_0\|^2_{\dot{H}^{1/2}} + \|w_{01}\|^2_{\dot{H}^{-1/2}} + \|w_{10}\|^2_{\dot{H}^{-1/2}}).$$

Additionally, the higher energy norms with which one defines the X^m topology for (u_0,u_1) are also bounded by this extension process, namely

$$\|(u_0,u_1)\|^2_{X^{m,C}} \le C_m(\|w_0\|^2_{\dot{H}^{m+1/2}} + \|w_{01}\|^2_{\dot{H}^{m-1/2}} + \|w_{10}\|^2_{\dot{H}^{m-1/2}}).$$

Proof. (of Theorem 3.2): Using the Plancherel identity, the $L^2(N)$ norm of $E(w)$ is

$$\begin{aligned}\|E(w)\|^2_{L^2(N)} &= \iint |\hat{w}(\xi)|^2\,\psi^2(\eta'/\left|\xi\right|)\frac{1}{|\xi|^2}d\eta' d\xi \\ &= \int \frac{1}{|\xi|}|\hat{w}(\xi)|^2\left(\int \psi^2(\eta'/\left|\xi\right|)\frac{1}{|\xi|}d\eta'\right)d\xi \\ &= \|\psi\|^2_{L^2[-1,1]}\,\|w\|^2_{\dot{H}^{-1/2}(M)}\,,\end{aligned}$$

since $\theta = \eta'/\left|\xi\right|$ and

$$\int \psi^2(\eta'/\left|\xi\right|)\frac{1}{|\xi|}d\eta' = \int_{-1}^{1}\psi^2(\theta)d\theta.$$

The identity extends to the Sobolev space $H^m(N)$; it suffices to calculate $\left\|\partial^m_{x_1}E(w)\right\|_{L^2}$ and $\left\|\partial^m_{y_2}E(w)\right\|_{L^2}$:

$$\begin{aligned}\left\|\partial^m_{x_1}E(w)\right\|_{L^2(N)} &= \iint |\hat{w}(\xi)|^2\,|\xi|^{2m}\,\psi^2(\eta'/\left|\xi\right|)\frac{1}{|\xi|^2}d\eta' d\xi \\ &= \int |\hat{w}(\xi)|^2\,|\xi|^{2m-1}\left(\int \psi^2(\eta'/\left|\xi\right|)\frac{1}{|\xi|}d\eta'\right)d\xi \\ &= \|\psi\|^2_{L^2[-1,1]}\,\|w\|^2_{\dot{H}^{m-1/2}(M)}\,.\end{aligned}$$

The second quantity is similar:

$$\begin{aligned}\left\|\partial_{y_2}^m E(w)\right\|_{L^2(N)} &= \iint |\hat{w}(\xi)|^2 |\xi|^{2m} \psi^2(\eta'/|\xi|)\frac{1}{|\xi|^2} |\eta'|^{2m} d\eta' d\xi \\ &= \int |\hat{w}(\xi)|^2 |\xi|^{2m-1} \left(\int \psi^2(\eta'/|\xi|) \left|\frac{\eta'}{|\xi|}\right|^{2m} \frac{1}{|\xi|} d\eta' \right) d\xi \\ &= C_m \|w\|^2_{\dot{H}^{m-1/2}(M)} ,\end{aligned}$$

where $C_m = \int\limits_{-1}^{1} \theta^{2m}\psi^2(\theta)d\theta$. The third and fourth estimates of the theorem involve $y_2 E(w)$, whose Fourier transform is

$$\hat{w}(\xi)\frac{1}{i}\partial_{\eta'}\chi(\eta'/|\xi|).$$

Measuring the L^2 norm of $y_2E(w)$,

$$\begin{aligned}\|y_2 E(w)\|^2_{L^2(N)} &= \iint |\hat{w}(\xi)|^2 \left| \frac{1}{i}\partial_\theta \psi(\eta'/|\xi|)\frac{1}{|\xi|^2}\right|^2 d\eta' d\xi \\ &= \int |\hat{w}(\xi)|^2 \frac{1}{|\xi|^3} \left(\int |\partial_\theta\psi(\eta'/|\xi|)|^2 \frac{1}{|\xi|} d\eta' \right) d\xi \\ &= \int |\hat{w}(\xi)|^2 \frac{1}{|\xi|^3} d\xi \left(\int\limits_{-1}^{1} |\partial_\theta \psi|^2 d\theta \right) \\ &= C\|w\|^2_{\dot{H}^{-3/2}(M)}\end{aligned}$$

with $C = \int\limits_{-1}^{1} |\partial_\theta\psi|^2 d\theta$. The calculations of the H^m norms of $y_2E(w)$ are similar. □

Proof. (of Theorem 3.3): Given initial data $(w_0, w_{01}, w_{10})(x_1)$ we are to give conditions under which the energy norm of the extension (u_0, u_1) is finite. First of all, the contribution to the energy given by u_1 is simply $\frac{1}{2}\|u_1\|^2_{L^2}$, hence by Theorem 3.2 it is bounded by $C\|w_{10}\|^2_{\dot{H}^{-1/2}}$. There are two contributions from u_0, which can be expressed using the Plancherel identity:

$$\iint \omega^2(\xi,\eta')\, |\hat{w}_0(\xi)|^2 \chi^2(\xi,\eta') d\eta' d\xi + \iint \omega^2(\xi,\eta')\, |\hat{w}_{01}(\xi)|^2 \left|\partial_{\eta'}\chi^2(\xi,\eta')\right|^2 d\eta' d\xi \ .$$

Using that $\chi(\xi,\eta') = \psi(\eta'/|\xi|)\frac{1}{|\xi|}$, we estimate these two integrals:

$$\begin{aligned}&\iint\limits_{\{|\eta'|<|\xi|\}} \left(|\xi|^2 - |\eta'|^2\right) |\hat{w}_0(\xi)|^2 \psi^2(\eta'/|\xi|)\frac{1}{|\xi|^2} d\eta' d\xi \\ &= \int |\hat{w}_0(\xi)|^2 \left[\int \left(|\xi| - \frac{|\eta'|^2}{|\xi|}\right) \psi^2(\eta'/|\xi|)\frac{1}{|\xi|} d\eta' \right] d\xi \\ &\le C\|w_0\|^2_{\dot{H}^{1/2}} .\end{aligned}$$

$$\begin{aligned}
&\iint\limits_{\{|\eta'|<|\xi|\}} \left(|\xi|^2 - |\eta'|^2\right) |\hat{w}_{01}(\xi)|^2 \psi^2(\eta'/|\xi|) \frac{1}{|\xi|^2} d\eta' d\xi \\
&= \int |\hat{w}_{01}(\xi)|^2 \left[\int_{\{|\eta'|<|\xi|\}} \left(\frac{1}{|\xi|} - \frac{|\eta'|^2}{|\xi|^3} \right) \psi^2(\eta'/|\xi|) \frac{1}{|\xi|} d\eta' \right] d\xi \\
&\leq C \|w_{01}\|^2_{\dot{H}^{-1/2}} .
\end{aligned}$$

□

(b) The extension problem for general spacelike data

We now consider the general problem of initial data given on a maximal spacelike hypersurface of dimension d_1, extending it to Cauchy data on a codimension one hypersurface. That is, for $(x, y) \in \mathbb{R}^{d_1}_x \times \mathbb{R}^{d_2}_y$,

$$M = \{y = 0\} \subseteq N = \{y_1 = 0\}.$$

Initial data on M take the form $w(x) = (w_0(x), w_\alpha(x))$ where a solution $u(x, y)$ of the field equation (1.2) is asked to satisfy

$$u(x, y) = w_0(x)$$

with its first derivatives normal to M satisfying

$$\partial^\alpha_y u(x, 0) = w_\alpha(x)$$

where $\alpha \in \mathbb{N}^{d_2}$ is the multi-index $\alpha = (\alpha_1, ..., \alpha_{d_2})$, $|\alpha| = 1$, such that only one $\alpha_j = 1$ and the rest are zero. The object is to extend $w(x)$ to Cauchy data on N while satisfying the constraints (2.9) to be in X^C. This Cauchy data satisfies

$$\begin{aligned}
u_0(x, 0) &= w_0(x) \\
u_{\alpha'}(x, 0) &= w_{0\alpha'}(x)
\end{aligned}$$

for $\alpha' = (\alpha_2, ..., \alpha_{d_2})$ and the first derivatives normal to N satisfy

$$\partial_{y_1} u(x, 0) = w_{10}(x) \ .$$

Following the construction given in section 3.1, define an extension operator

$$E(w)(x, y') := \frac{1}{\sqrt{2\pi}^{d_1+d_2-1}} \iint \hat{w}(\xi)\chi(\xi, \eta') e^{i\xi\cdot x} e^{i\eta'\cdot y'} d\xi d\eta'$$

where the kernel function is even in η and satisfies

$$\frac{1}{\sqrt{2\pi}^{d_2-1}} \int \chi(\xi, \eta') d\eta' = 1 \ .$$

To satisfy the constraint that $E(w) \in X^C$, we ask that $\mathrm{supp}(\chi(\xi, \eta')) \subseteq \{(\xi, \eta') : |\eta'| < |\xi|\}$. Such kernel functions are readily constructed (they are far from being uniquely determined). For example, a variant of our construction of section 3.1 is

based on choice of a C_0^∞ function $\psi(\theta) \geq 0$, with $\mathrm{supp}(\psi) \subseteq B_1(0)$, the ball of radius one. Then define

$$\chi(\xi, \eta') = \psi(\eta' / |\xi|) \frac{1}{|\xi|^{d_2 - 1}} \ .$$

We note that χ is even in η if $\psi(\theta)$ is even, and that

$$\begin{aligned} \frac{1}{\sqrt{2\pi}^{d_2-1}} \int \chi(\xi, \eta') d\eta' &= \frac{1}{\sqrt{2\pi}^{d_2-1}} \int \psi(\eta' / |\xi|) \frac{1}{|\xi|^{d_2-1}} d\eta' \\ &= \frac{1}{\sqrt{2\pi}^{d_2-1}} \int \psi(\theta) d\theta \ . \end{aligned}$$

This is normalized to one by choice of ψ.

Theorem 3.4. *The extension operator E is bounded on the following function spaces:*

$$\begin{aligned} E : \dot{H}^{\frac{1-d_2}{2}}(M) &\to L^2(N) \\ \dot{H}^{\frac{1-d_2}{2}}(M) \cap H^{m+\frac{1-d_2}{2}}(M) &\to H^m(N) \ , \end{aligned}$$

with m the exponent of Sobolev regularity, and

$$\begin{aligned} y'E : \dot{H}^{\frac{-(1+d_2)}{2}}(M) &\to L^2(N) \\ \dot{H}^{\frac{-(1+d_2)}{2}}(M) \cap H^{m-\frac{1+d_2}{2}}(M) &\to H^m(N) \ . \end{aligned}$$

Using the extension operator E, the vector function $w(x) = (w_0(x), w_\alpha(x))$ extends to Cauchy data on N as follows:

$$\begin{aligned} u_0(x, y') &:= E(w_0)(x, y') + \sum_{|\alpha'|=1} y^{\alpha'} \cdot E(w_{0\alpha'})(x, y') \\ u_1(x, y') &:= E(w_{10})(x, y') \ . \end{aligned}$$

This is seen to extend the initial data $w(x)$ in the required way, and in addition it satisfies the constraint that $(u_0, u_1) \in X^C$. However, measuring the functions (u_0, u_1) in the energy norm is more appropriate for the Cauchy problem, hence we also state estimates in this setting.

Theorem 3.5. *Given $w_0 \in \dot{H}(M)$ and $w_\alpha \in \dot{H}(M)$, the energy norm of the extension*

$$(u_0, u_1) = (E(w_0) + y^{\alpha'} \cdot E(w_{(0,\alpha')}), E(w_{10}))$$

is finite; indeed

$$\|(u_0, u_1)\|^2_{X^C} \leq C(\|w_0\|^2_{\dot{H}^{\frac{3-d_2}{2}}} + \|w_{0\alpha'}\|^2_{\dot{H}^{\frac{1-d_2}{2}}} + \|w_{10}\|^2_{\dot{H}^{\frac{1-d_2}{2}}}) \ .$$

The proofs of Theorems 3.4 and 3.5 are similar to the proofs of Theorems 3.2 and 3.3, to which we refer the reader.

(c) *The extension problem for mixed spacelike and timelike data*

As a final case, we consider the extension problem for initial data on a lower dimensional hypersurface M of *mixed* signature. Given zero'th and first normal derivatives of a solution $u(x,y)$ on M, the object is to extend this data to the codimension one hypersurface $N = \{y_1 = 0\}$ in such a way that the constraint for well-posedness is satisfied. This is not always possible for arbitrary data $w = (w_0, w_\alpha)$ posed on M, due to analogous lower dimensional constraints on M. But it is possible, along with attendant Sobolev bounds on the extended functions, in most cases. This situation will be explained below.

To set the notation, we consider spacelike and timelike coordinates on M to be $(\tilde{x}, \tilde{y}) \in \mathbb{R}^{p_1} \times \mathbb{R}^{p_2}$, with their Fourier transform variables denoted $(\tilde{\xi}, \tilde{\eta}) \in \mathbb{R}^{p_1} \times \mathbb{R}^{p_2}$. The complementary variables will be denoted $(x'', y'') \in \mathbb{R}^{d_1 - p_1} \times \mathbb{R}^{d_2 - p_2 - 1}$ and $(\xi'', \eta'') \in \mathbb{R}^{d_1 - p_1} \times \mathbb{R}^{d_2 - p_2 - 1}$, so that coordinates on N are $(x, y') = (\tilde{x}, x'', \tilde{y}, y'')$. The evolution variable remains y_1.

Initial data for a solution $u(x,y)$ is given on N, which is expressed in the form $(u, \partial_{x''}^{\alpha''} u, \partial_{y_1}^{\beta_1} u, \partial_{y''}^{\beta''} u)(\tilde{x}, \tilde{y}, 0, 0) = (w_0, w_{\alpha''}, w_{\beta_1}, w_{\beta''})(\tilde{x}, \tilde{y})$, where $\alpha'' = (\alpha_{p_1+1}, \ldots, \alpha_{d_1})$, $\beta'' = (\beta_{p_2+1}, \ldots, \beta_{d_2})$ are multi-indices such that $|\alpha''| + |\beta''| + |\beta_1| = 1$. The idea is the same as in sections 3.1 and 3.2, namely to extend $(w_0, w_{\alpha''}, w_{\beta_1}, w_{\beta''})$ to constraint-satisfying Cauchy data on N in such a way that a solution $u(x,y) = u(\tilde{x}, x'', \tilde{y}, y'')$ to the field equation (1.2) satisfies

$$u(\tilde{x}, 0, 0, \tilde{y}, 0) = w_0(\tilde{x}, \tilde{y})$$

and

$$\partial_{y_1} u(\tilde{x}, 0, 0, \tilde{y}, 0) = w_{0\beta_1}(\tilde{x}, \tilde{y}) \ ,$$

as well as the compatibility conditions

$$\begin{aligned} \partial_{x''}^{\alpha''} u(\tilde{x}, 0, 0, \tilde{y}, 0) &= w_{(\alpha'', 0)}(\tilde{x}, \tilde{y}) \\ \partial_{y''}^{\beta''} u(\tilde{x}, 0, 0, \tilde{y}, 0) &= w_{(0, \beta'')}(\tilde{x}, \tilde{y}) \end{aligned}$$

The existence of such an extension follows as in Theorems 3.4 and 3.5 from the construction of an extension operator E with certain boundedness properties on appropriate Sobolev spaces. We will focus our analysis therefore on the extension operators.

Again following section 3.1, define an extension operator

$$E(w)(x, y') = \frac{1}{\sqrt{2\pi}^{d_1 + d_2 - 1}} \iint \chi(\tilde{\xi}, \xi'', \tilde{\eta}, \eta'') d\xi'' d\eta'' = 1.$$

Furthermore, to satisfy the constraint that $E(w) \in X^C$ for arbitrary data w, we ask that

$$\operatorname{supp}(\chi(\xi, \eta')) \subseteq \left\{ (\xi, \eta') : |\eta'|^2 < |\xi|^2 \right\} := R_1.$$

These two conditions are always satisfiable, except in the case $\xi'' = \{0\}$, meaning that $d_1 = p_1$ and the extension subspace $\{(\xi'', \eta'')\}$ is purely timelike.

It is to be expected that the constraint induces a restriction on the data $w(\tilde{\xi},\tilde{\eta})$ in the vicinity of the "lightcone" $\left\{\left|\tilde{\xi}\right| = |\tilde{\eta}|\right\} \subseteq \hat{M}$. Subdivide $\hat{M}$ into two sets,

$$\tilde{R}_1 := \left\{(\tilde{\xi},\tilde{\eta}) \in \hat{M} : |\tilde{\eta}| \leq \left|\tilde{\xi}\right|\right\}$$
$$\tilde{R}_2 := \left\{(\tilde{\xi},\tilde{\eta}) \in \hat{M} : |\tilde{\eta}| > \left|\tilde{\xi}\right|\right\}.$$

The orthogonal projections onto functions supported in $\tilde{R}_1$, $\tilde{R}_2$ respectively, are denoted π_1 and π_2. We use standard Sobolev spaces to quantify data supported in $\tilde{R}_1$, namely

$$H^r = \left\{ w(\tilde{x},\tilde{y}) \in \mathrm{range}(\pi_1) : \|w\|^2_{H^r} = \iint\limits_{\tilde{R}_1} \left|\hat{w}(\tilde{\xi},\tilde{\eta})\right|^2 (\left|\tilde{\xi}\right|^2 + |\tilde{\eta}|^2)^n d\tilde{\xi} d\tilde{\eta} < +\infty \right\} .$$

Over $\tilde{R}_2$ we use a modified form of Sobolev norm which is given by

$$K^r = \left\{ w(\tilde{x},\tilde{y}) \in \mathrm{range}(\pi_2) : \|w\|^2_{K^r} = \iint\limits_{\tilde{R}_2} \left|\hat{w}(\tilde{\xi},\tilde{\eta})\right|^2 \frac{(\left|\tilde{\xi}\right|^2 + |\tilde{\eta}|^2)^r}{(|\tilde{\eta}|^2 - \left|\tilde{\xi}\right|^2)^{\frac{1}{2}e_0}} d\tilde{\xi} d\tilde{\eta} < +\infty \right\}.$$

where

$$e_0 := d_1 + d_2 - (p_1 + p_2) - 1.$$

We note that in the case where $d_1 = p_1$, $\tilde{R}_1 = \{0\}$ and $K^n = H^{r-\frac{r}{2}(d_2-p_2-1)}$. More generally, define

$$K^r_s = \left\{ w(\tilde{x},\tilde{y}) \in \mathrm{range}(\pi_2) : \|w\|^2_{K^r_s} = \iint\limits_{\tilde{R}_2} \left|\hat{w}(\tilde{\xi},\tilde{\eta})\right|^2 \frac{(\left|\tilde{\xi}\right|^2 + |\tilde{\eta}|^2)^r}{(|\tilde{\eta}|^2 - \left|\tilde{\xi}\right|^2)^{\frac{1}{2}e_0+s}} d\tilde{\xi} d\tilde{\eta} < +\infty \right\}$$

Decompose an arbitrary function $w = \pi_1 w + \pi_2 w$, so that its components possess Fourier support in $\tilde{R}_1$ and $\tilde{R}_2$ respectively.

Theorem 3.6. *If $d_1 > p_1$ then there is a choice of kernel χ (indeed there are many such choices) such that $u = E(w)$ satisfies*

$$\|u\|^2_{L^2} \leq C(\|\pi_1 w\|^2_{H^{-\frac{1}{2}(e_0)}} + \|\pi_2 w\|^2_{K^0}).$$

Higher Sobolev norms of $u = E(w)$ are bounded as follows

$$\|u\|^2_{H^r} \leq C_r(\|\pi_1 w\|^2_{H^{r-\frac{1}{2}(e_0)}} + \|\pi_2 w\|^2_{K^r} .$$

In case $d_1 = p_1$, it is not possible to extend arbitrary data to a function $u = E(w)$ which satisfies the constraint $\mathrm{supp}(\hat{u}(\xi,\eta')) \subseteq R_1$*. However, if initially* $\mathrm{supp}(\hat{w}(\xi,\eta')) \subseteq \tilde{R}_1$ *(i.e., $w = \pi_2 w$), then such an extension is possible, and we have, for $u = E(w)$,*

$$\|u\|^2_{L^2} \leq C \|w\|^2_{K^0} \;,$$
$$\|u\|^2_{H^r} \leq C_r \|w\|^2_{K^r} \;.$$

Proof. The proof of Theorem 3.6 depends upon the construction of a kernel $\chi(\tilde{\xi},\tilde{\eta},\xi'',\eta'')$ with satisfactory properties. This construction is slightly different in the two different regions of Fourier space

$$\tilde{R}_1 := \left\{(\tilde{\xi},\tilde{\eta}) : |\tilde{\eta}| \le \left|\tilde{\xi}\right|\right\} \text{ and } \tilde{R}_2 := \left\{(\tilde{\xi},\tilde{\eta}) : |\tilde{\eta}| > \left|\tilde{\xi}\right|\right\}$$

where we note that the region $\tilde{R}_2$ contains the data which would lead to an ill-posed initial value problem if M were considered itself as a codimension one hypersurface.

To extend data posed on region $\tilde{R}_1$, define

$$\chi_1(\tilde{\xi},\tilde{\eta},\xi'',\eta'') := \psi_1\Big(\frac{\xi''}{(\left|\tilde{\xi}\right|^2+|\tilde{\eta}|^2)^{\frac{1}{2}}},\frac{\eta''}{(\left|\tilde{\xi}\right|^2+|\tilde{\eta}|^2)^{\frac{1}{2}}}\Big)\cdot\frac{1}{(\left|\tilde{\xi}\right|^2+|\tilde{\eta}|^2)^{\frac{1}{2}e_0}},$$

where $\psi_1(\theta_1,\theta_2)$ is a C_0^∞ function of $(d_1-p_1)\times(d_2-p_2-1)$ variables, respectively, with support in the set $|\theta_2|<|\theta_1|$. Therefore χ_1 has support in the set

$$\frac{\xi''}{(\left|\tilde{\xi}\right|^2+|\tilde{\eta}|^2)^{\frac{1}{2}}} \ge \frac{\eta''}{(\left|\tilde{\xi}\right|^2+|\tilde{\eta}|^2)^{\frac{1}{2}}}$$

implying that

$$|\eta'|^2 = |\tilde{\eta}|^2 + \left|\eta''\right|^2 < \left|\tilde{\xi}\right|^2 + |\xi''|^2 = |\xi|^2 .$$

This is the appropriate region of support from functions $v = E(w)$ to lie in the constraint-satisfying subspace of $L^2(N)$. In order that E be an extension operator, we furthermore require that

$$\begin{aligned}
\sqrt{2\pi}^{d_1+d_2-1} &= \iint \chi_1(\tilde{\xi},\tilde{\eta},\xi'',\eta'')d\xi''d\eta'' \\
&= \iint \psi_1\Big(\frac{\xi''}{(\left|\tilde{\xi}\right|^2+|\tilde{\eta}|^2)^{\frac{1}{2}}},\frac{\eta''}{(\left|\tilde{\xi}\right|^2+|\tilde{\eta}|^2)^{\frac{1}{2}}}\Big)\cdot\frac{1}{(\left|\tilde{\xi}\right|^2+|\tilde{\eta}|^2)^{\frac{1}{2}e_0}}d\xi''d\eta'' \\
&= \iint \psi_1(\theta_1,\theta_2)d\theta_1 d\theta_2.
\end{aligned}$$

Asking that this latter integral equal the normalizing constant $\sqrt{2\pi}^{d_1+d_2-1}$, and asking for ψ_1 to be even in its variables (θ_1,θ_2) gives an acceptable kernel for the extension operator. We note again that this choice of kernel is highly nonunique.

On the region $\tilde{R}_2 = \left\{(\tilde{\xi},\tilde{\eta}) \in \hat{M} : |\tilde{\eta}| > \left|\tilde{\xi}\right|\right\}$, we can also attempt a construction of our extension operator. By itself, this region would give rise to data in $L^2(M)$ for which the Cauchy problem of mixed type is ill-posed. The extension operator will nonetheless come up with data $u = E(w)$ for which the well-posedness constraint is satisfied, if this is possible. That is, as long as $d_1 > p_1$, so that $\{\xi''\}$ is not restricted to the zero-dimensional vector space, extensions can be found in a way that the default in satisfying the constraint caused by the fact that $\left|\tilde{\xi}\right| < |\tilde{\eta}|$ can be made up with a choice of large $|\xi''|$. In practice, we will build $\chi_2(\tilde{\xi},\tilde{\eta},\xi'',\eta'')$ so that its support is in the regions

$$\left\{|\tilde{\eta}| > \left|\tilde{\xi}\right|\right\} = \tilde{R}_2$$

as well as

$$\left\{|\tilde{\eta}|^2 + |\eta''|^2 < \left|\tilde{\xi}\right|^2 + |\xi''|^2\right\};$$

implying that $0 \le (|\tilde{\eta}|^2 - \left|\tilde{\xi}\right|^2) + (|\eta''|^2 + |\xi''|^2)$. Thus we require $d_1 > p_1$. Following the above examples, assume that $d_1 > p_1$ and set

$$\chi_2(\tilde{\xi}, \tilde{\eta}, \xi'', \eta'') := \psi_2\Big(\frac{\xi''}{(|\tilde{\eta}|^2 - \left|\tilde{\xi}\right|^2)^{\frac{1}{2}}}, \frac{\eta''}{(|\tilde{\eta}|^2 - \left|\tilde{\xi}\right|^2)^{\frac{1}{2}}}\Big) \cdot \frac{1}{(|\tilde{\eta}|^2 - \left|\tilde{\xi}\right|^2)^{\frac{1}{2}e_0}}$$

for $(\tilde{\xi}, \tilde{\eta}) \in \tilde{R}_2$. Let $\psi_2(\theta_1, \theta_2)$ be a C_0^∞ function of $e_0 = d_1 + d_2 - (p_1 + p_2) - 1$ variables, as before and require that

$$\begin{aligned}\int \psi_2(\theta_1, \theta_2) d\theta_1 d\theta_2 &= \int \psi_2\Big(\frac{\xi''}{(|\tilde{\eta}|^2 - \left|\tilde{\xi}\right|^2)^{\frac{1}{2}}}, \frac{\eta''}{(|\tilde{\eta}|^2 - \left|\tilde{\xi}\right|^2)^{\frac{1}{2}}}\Big) \cdot \frac{1}{(|\tilde{\eta}|^2 - \left|\tilde{\xi}\right|^2)^{\frac{1}{2}e_0}} d\xi'' d\eta'' \\ &= \sqrt{2\pi}^{e_0}.\end{aligned}$$

Furthermore, ask that $\psi(\theta_1, \theta_2)$ be even in (θ_1, θ_2). Finally ask that the support of $\psi(\theta_1, \theta_2)$ be in the set

$$\left\{(\theta_1, \theta_2) : \theta_1^2 - \theta_2^2 > 1\right\}.$$

Such requirements are satisfied by many possible choices of ψ. In doing so, we arrive at a satisfactory kernel of an extension operator E with the property that all functions $u = E(w)$ in its range have Fourier support satisfying $\mathrm{supp}(\hat{u}) \le \left\{|\eta'|^2 < |\xi|^2\right\}$. The singularities introduced at the boundaries of the lightcone $\left\{|\tilde{\eta}| = \left|\tilde{\xi}\right|\right\} \subseteq \hat{M}$ by the kernel χ_2 impose more severe constraints on the functions w that are permitted in the domain of the operator E; this is the origin of the somewhat unusual requirements on functions $w(\tilde{x}, \tilde{y})$ from which we can reasonably draw our data. The Sobolev estimates of the proof are similar to those of Theorems 3.4 and 3.5 and we leave the details to the reader. □

Finally, we show that a sufficiently large class of data $(w_0, w_{(\alpha'',0)}, w_{(0,\beta'')}, w_{\beta_1})$ on M extends to Cauchy data on the hypersurface N which is both of finite energy and satisfies the constraint. This extension is given by

$$\begin{aligned} u_0(x, y') &= E(w_0)(x, y') + \sum_{|\alpha''|=1} x''^{\alpha''} E(w_{(\alpha'',0)})(x, y') + \sum_{|\beta''|=1} y''^{\beta''} E(w_{(0,\beta'')})(x, y') \\ u_1(x, y') &= E(w_{(0,\beta_1)})(x, y'). \end{aligned} \tag{3.2}$$

By design, this Cauchy data satisfies the constraint, that is, $(u_0, u_1) \in X^C$, the center manifold. As before, its restriction to M reduces to the data $(w_0, w_{(\alpha'',0)}, w_{(0,\beta'')})(\tilde{x}, \tilde{y}, 0)$. The only remaining task is to show that its energy norm is finite. Recall that in

this context the energy norm is

$$
\begin{aligned}
H(u_0, u_1) &= \frac{1}{2} \iint_N |u_1|^2 + |\nabla_x u_0|^2 - |\nabla_{y'} u_0|^2 \, dx dy' \\
&= \frac{1}{2} \iint_N |\hat{u}_1(\xi, \eta')|^2 + (|\xi|^2 - |\eta'|^2) \hat{u}_0(\xi, \eta') d\xi d\eta' .
\end{aligned}
$$

To show that this energy is finite for the extension (3.2), we use the results of Theorem 3.6.

Theorem 3.7. *Given data* $(w_0, w_{(\alpha'',0)}, w_{(0,\beta'')}, w_{\beta_1})$ *on* M *with* $|\alpha''| = |\beta''| = 1$, *suppose that*

$$
\|\pi_1 w_0\|_{H^{e_0+1}} + \sum_{|\alpha''|=1} \left\|\pi_1 w_{(\alpha'',0)}\right\|_{H^{e_0+1}} + \sum_{|\beta''|=1} \left\|\pi_1 w_{(0,\beta'')}\right\|_{H^{e_0+1}} < +\infty \tag{3.3}
$$

$$
\|\pi_2 w_0\|_{K^1} + \sum_{|\alpha''|=1} \left\|\pi_2 w_{(\alpha'',0)}\right\|_{K^1} + \sum_{|\beta''|=1} \left\|\pi_2 w_{(0,\beta'')}\right\|_{K^1} < +\infty \tag{3.4}
$$

and

$$
\|\pi_1 w_{\beta_1}\|_{H^{e_0}} + \|\pi_2 w_{\beta_1}\|_{K^0} + \infty .
$$

Then the extension (u_0, u_1) *given by expression (3.2) has finite energy and lies in the center subspace* X^C. *If* $d_1 = p_1$, *then we have to ask that* $\pi_2 w_\gamma = 0$ *in the above statement, for all multi-indices* γ *in question.*

Proof. Estimates on the contributions of w_0 to u_0 follow immediately from Theorem 3.6, as do the estimates for $u_1 = E(w_{\beta_1})$. Therefore we only have to consider contributions in one of the two possible forms:

$$
x''^{\alpha''} E(w_{(\alpha'',0)}) \qquad |\alpha''| = 1
$$

or

$$
y''^{\beta''} E(w_{(0,\beta'')}) \qquad |\beta''| = 1 \ .
$$

The energy norm includes the quantities $\left\|x''^{\alpha''} E(w_{(\alpha'',0)})\right\|_{H^1}$ and $\left\|y''^{\beta''} E(w_{(0,\beta'')})\right\|_{H^1}$; since the estimates are similar we will give a sketch of one of them.

$$
\begin{aligned}
&\left\|x''^{\alpha''} E(w_{(\alpha'',0)})\right\|_{H^1}^2 = \left\|\frac{1}{i} \partial_{\xi''}^{a''} \widehat{E(w_{(\alpha'',0)})} (|\xi|^2 + |\eta'|^2)^{\frac{1}{2}}\right\|_{L^2}^2 \\
&\leq \iiiint \Big[\partial_{\xi''}^{a''} \psi_1 \Big(\frac{\xi''}{(\left|\tilde{\xi}\right|^2 + |\tilde{\eta}|^2)^{\frac{1}{2}}}, \frac{\eta''}{(\left|\tilde{\xi}\right|^2 + |\tilde{\eta}|^2)^{\frac{1}{2}}} \Big) \cdot \frac{1}{(\left|\tilde{\xi}\right|^2 + |\tilde{\eta}|^2)^{\frac{1}{2} e_0}} \Big]^2 \left|\pi_1 \widehat{w_{(\alpha'',0)}}(\tilde{\xi}, \tilde{\eta})\right|^2 \\
&+ \Big[\partial_{\xi''}^{a''} \psi_2 \Big(\frac{\xi''}{(|\tilde{\eta}|^2 - \left|\tilde{\xi}\right|^2)^{\frac{1}{2}}}, \frac{\eta''}{(|\tilde{\eta}|^2 - \left|\tilde{\xi}\right|^2)^{\frac{1}{2}}} \Big) \cdot \frac{1}{(|\tilde{\eta}|^2 - \left|\tilde{\xi}\right|^2)^{\frac{1}{2} e_0}} \Big]^2 \left|\pi_2 \widehat{w_{(\alpha'',0)}}(\tilde{\xi}, \tilde{\eta})\right|^2 d\tilde{\xi} d\tilde{\eta} d\xi'' d\eta'' .
\end{aligned}
$$

The ξ''-derivative introduces one extra factor of $(\left|\tilde{\xi}\right|^2 + |\tilde{\eta}|^2)$, respectively $(|\tilde{\eta}|^2 - \left|\tilde{\xi}\right|^2)$, into the denominator. The integral over (ξ'', η'') gives a constant, depending upon ψ_1 and ψ_2, as a bound, while the resulting integral over the variable $(\tilde{\xi}, \tilde{\eta})$ is bounded by the H^{1-e_0} norm (respectively, the K_1^1 norm) of $w_{(\alpha'',0)}$. This finishes the proof. $\square$

4. Failure of uniqueness in higher codimension

The question addressed in this section is the uniqueness of solutions with prescribed initial data on a hypersurface M of codimension greater than one. This is a nontrivial issue if one requires that solutions exist globally in space-time, which has been the focus of the analysis in the preceding sections. In section 3 we showed that initial data consisting of the values of the solution $u(x, y)$ and its first normal derivatives on M, through a procedure of extension, give rise to constraint-satisfying Cauchy data on a codimension-one hypersurface N. These extensions are highly nonunique, and therefore so are the resulting global solutions.

We now raise the question whether prescribing an arbitrarily large but finite number of normal derivatives on M, as well as insisting upon global solutions, would remedy the nonuniqueness. This data should satisfy the compatibility conditions implied by the commuting of mixed partial derivatives and by equation (1.2). Given Courant's classic result (1962) in the case of purely timelike M, that data given in any ε-tubular neighborhood of M within N determine solutions uniquely in the C^2 category, one might think that specifying additional data for $u(x, y)$ on M would suffice. In fact, if one specifies any finite number of derivatives of u on M it does not.

Theorem 4.1. *Given k, there exist constraint-satisfying data u_0, u_1 on N which vanish to order k on M.*

Therefore, there exists a globally defined solution $u(x, y)$ which has initial data $u(x, y) = u_0$, $\partial_{y_1} u(x, y) = u_1$ on N, which vanishes to order k on M. Hence any other solution $v(x, y)$ which takes on specified data on M up to k-many derivatives may be changed by adding this solution u to it, without changing its initial data.

Proof. We follow a construction that was used for the extension operators of section 3. Let $\chi_3(\xi, \eta')$ be a Schwartz class function with support in the set $\left\{|\eta'|^2 < |\xi|^2\right\} \subseteq \hat{N}$. Its Fourier restriction to $\hat{M}$, given by

$$\iint \chi_3(\tilde{\xi}, \tilde{\eta}, \xi'', \eta'') d\xi'' d\eta'' = \mu(\tilde{\xi}, \tilde{\eta})$$

is in Schwartz class in $\hat{M}$. Because of the support of χ_3,

$$v(x, y') = (\mathcal{F}^{-1}\chi_3)(x, y')$$

satisfies the constraint. While v may be nonzero on M, as may its derivatives, it is the case that for homogeneous polynomials $p_{k+1}(x'', y'')$ of degree $k + 1$, the function $p_{k+1}(x'', y'')v(x, y')$ on N vanishes on M to at least order k. Furthermore $p_{k+1}v$ satisfies the constraint. Indeed,

$$(\mathcal{F}p_{k+1}v)(\xi, \eta') = p_{k+1}(\frac{1}{i}\partial_{\xi''}, \frac{1}{i}\partial_{\eta''})\chi_3(\xi, \eta'),$$

and differential operators do not affect the support. Set data $u_0(x, y') = (p_{k+1}v)(x, y')$ and $u_1 = 0$, and solve equation (1.2). Because this data satisfies the constraint, the solution $u(x, y)$ is global. Because of the properties of the initial data, all x and y' derivatives of $u(x, y)$ vanish on M. Because $u_1 = 0$ and u itself satisfies equation (1.2), all y_1 derivatives up to order k as well as any mixed derivatives also vanish. $\square$

5. A variant of a uniqueness theorem of Courant

Courant (1962) gives a uniqueness result for the ultrahyperbolic equation with data posed on a hypersurface of mixed signature, which in our notation states that, among C^2 solutions, initial values of $u(x,0,y')$ and $\partial_{y_1} u(x,0,y')$ prescribed in the set in the Cauchy hypersurface M given by

$$\sum_{\ell=1}^{d_1} (x_\ell - x_\ell^0)^2 \le a^2 \ , \qquad \sum_{\ell=2}^{d_2} (y_\ell - y_\ell^0)^2 \le \varepsilon^2 \tag{5.1}$$

will determine *a priori* the values of the data on the larger set

$$\left\{ (x,y') \in M \ : \ \sqrt{\sum_{\ell=1}^{d_1} (x_\ell - x_\ell^0)^2} + \sqrt{\sum_{\ell=2}^{d_2} (y_\ell - y_\ell^0)^2} \le a \right\} \ . \tag{5.2}$$

Furthermore the solution is determined uniquely in the space-time region

$$\left\{ (x,y) \in \mathbb{R}^{d_1+d_2} \ : \ \sqrt{\sum_{\ell=1}^{d_1} (x_\ell - x_\ell^0)^2} + \sqrt{\sum_{\ell=1}^{d_2} (y_\ell - y_\ell^0)^2} \le a \right\} \ . \tag{5.3}$$

Courant's proof of this fact uses the Asgeirsson mean value theorem in a fundamental way.

The key implication from our point of view is that data on an arbitrarily small cylindrical subset of M, plus the stipulation of C^2 regularity, determine the data and indeed the solution on much larger sets of M and of space-time, respectively. In turn, knowledge of the data in a small cylinder determines the values of all of its derivatives on $N = \{(x,y) \ : \ y = 0\}$ (if the data are smooth). This contrasts to the case discussed in section 4, in which it is shown that specification of a possibly large but finite number of derivatives does not lead to unique solutions, even when the constraint is imposed and the resulting solutions are globally defined and smooth.

In this section we give a version of the above theorem of Courant, for data posed in ellipsoidal domains in the Cauchy hypersurface M, which are localized near the $\{y' = 0\}$ coordinate axis (or any translate thereof). Our proof of this result is based on the Holmgren–John theorem (John 1982), and therefore remains true under perturbations to the equation. Thus it is a robust generalization of the Courant result, which being based on Asgeirsson's theorem is true only for precisely the ultrahyperbolic equation.

Theorem 5.1. *Let $\varepsilon > 0$ and define the ellipsoid $Z_\varepsilon \subseteq M$ by*

$$Z_\varepsilon = \{(x,y) \ : \ y_1 = 0 \ , |x|^2 + \frac{|y'|^2}{\varepsilon^2} < 1\} \ , \qquad 0 < \varepsilon \le 1 \ . \tag{5.4}$$

A C^2 solution to (1.2) *whose Cauchy data vanishes on Z_ε must necessarily vanish on the set*

$$D = \{(x,y) \in \mathbb{R}^{d_1+d_2} \ : \ |x| + |y| < 1\}$$

and in particular its Cauchy data along with all derivatives must vanish on the subset of the Cauchy hypersurface given by $\{(x,y') \in M \ : \ |x| + |y'| < 1\}$.

Proof. Define $R_\varepsilon(w)$ to be the cone over Z_ε with vertex $v = (0, w_1, w') \in \{(x,y) : x = 0\}$. We will show that for any $w = (w_1, w')$ with $|w| \le 1$ (namely the unit sphere in $\mathbb{R}^{d_2}$), the region between the cone $R_\varepsilon(w)$ and the ellipsoid Z_ε is a region of determinacy for the ultrahyperbolic equation. The closure of the envelope of such ellipsoidal cones includes the region D; in fact it is slightly larger. The result will follow accordingly.

For a given $R_\varepsilon(w)$, the Holmgren–John theorem is based upon the construction of an analytic family of noncharacteristic hypersurfaces S_λ with which to sweep the region between Z_ε and $R_\varepsilon(w)$. Taking the case of the vertex $v = (0, w)$ with $w = e_1 := (1, 0)$, define

$$S_\lambda := \{(x,y) \ : \ (1 - y_1)^2 - \big(|x|^2 + \frac{|y'|^2}{\varepsilon^2}\big) = -\lambda\}$$

with $-1 \le \lambda \le 0$. The normal to S_λ is $N_\lambda = -2(x, (1-y_1), y'/\varepsilon^2)^T$, so that the characteristic form calculated on N_λ is

$$\frac{1}{4} N_\lambda^T \begin{pmatrix} -I_{d_1 \times d_1} & 0 \\ 0 & I_{d_2 \times d_2} \end{pmatrix} N_\lambda = \frac{1}{4}\big(-|x|^2 + (1-y_1)^2 + |y'|^2/\varepsilon^2\big) \ .$$

Taking into account that $(x, y_1, y') \in S_\lambda$ and solving for $(1 - y_1)^2$,

$$\frac{1}{4} N_\lambda^T \begin{pmatrix} -I_{d_1 \times d_1} & 0 \\ 0 & I_{d_2 \times d_2} \end{pmatrix} N_\lambda = \frac{1}{4}\Big(\big(\frac{1+\varepsilon^2}{\varepsilon^4}\big)|y'|^2 - \lambda\Big) \ .$$

Recalling that $\lambda < 0$ (except in the limiting case $S_\lambda \to R_\varepsilon$) observe that this family of hyperboloids constitutes a noncharacteristic analytic family which sweeps the region between Z_ε and $R_\varepsilon(e_1)$. Thus the Holmgren–John uniqueness theorem applies, and this region is a region of determinacy for the ultrahyperbolic equation (1.2).

We have already achieved the analogue of the statement (5.3) of Courant. Namely, given the values of a C^2 solution $u(x,y)$ to (1.2) in the space-time ellipsoid

$$W_\varepsilon := \{(x,y) \ : \ |x|^2 + \frac{|y|^2}{\varepsilon^2} < 1\} \ ,$$

we may slice it with a hyperplane which contains the x-coordinate axes but which is otherwise arbitrarily oriented in y, to determine a possible Z_ε, which in turn determines the solution over the larger conical region R_ε with base Z_ε. All of these regions have been shown to be domains of determinacy. Their union contains the set $D = \{(x,y) \ : \ |x| + |y| < 1\}$. Therefore if a solution vanishes in W_ε it must also vanish in D.

Returning to the problem of the domain of determinacy of the set $Z_\varepsilon \subseteq M$, we generalize the above construction to any $w \in \mathbb{R}^{d_2}$ with $|w| = 1$. Let $w = Re_1$ for $e_1 = (1, 0 \dots)$, where R is an orthogonal matrix. Changing variables to $z = Ry$ and using a symmetric matrix Q of signature $(-, + \dots)$, an analytic family of hyperboloids is given by

$$S_\lambda(w) := \{(x,z) \ : \ |x|^2 + \langle (z - e_1), Q(z - e_1)\rangle = \lambda\} \ ,$$

where the Euclidean inner product is given by $\langle\cdot,\cdot\rangle$. The matrix Q is to be chosen so that the intersections of the hyperboloids $S_\lambda(w)$ with the hypersurface M lie in Z_ε, and sweeps it as λ is varied.

At this point we may assume without loss of generality that $w = (w_1, w') = (w_1, w_2, 0\dots)$, whereupon Q may be chosen such that

$$Q = \begin{pmatrix} Q_2 & 0 \\ 0 & \frac{1}{\varepsilon^2} I'' \end{pmatrix} , \qquad Q_2^T = Q_2 \ ,$$

for Q_2 a 2×2 symmetric matrix with signature $(-,+)$. Furthermore, the above rotation is then set to be

$$R = \begin{pmatrix} R_2 & 0 \\ 0 & I'' \end{pmatrix} , \qquad R_2 = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} .$$

In $y-$coordinates the hyperboloid family is expressed

$$S_\lambda(w) := \{(x,y) \ : \ |x|^2 + \langle (y-w), R^T Q R (y-w)\rangle = \lambda\} \ ,$$

and the stipulation is that $S_0(w)$ should intersect the hypersurface M in the original ellipsoid Z_ε. This imposes the condition that

$$|x|^2 + \langle (x,0,y'), R^T Q R (x,0,y')\rangle := |x|^2 + \langle (x,0,y'), B(x,0,y')\rangle = |x|^2 + \frac{1}{\varepsilon^2}|y'|^2 \ ,$$

where B_2 is the upper left-hand 2×2 block of the matrix B. Therefore one finds the matrix elements of B_2

$$b_{11} = -\frac{\varepsilon^2 - \sin^2(\theta)}{\varepsilon\cos^2(\theta)} \ , \quad b_{12} = -\frac{\tan(\theta)}{\varepsilon^2} \ , \quad b_{22} = \frac{1}{\varepsilon^2} \ ,$$

and furthermore, the 2×2 matrix Q_2 is

$$Q_2 = \begin{pmatrix} -1 & \tan(\theta) \\ \tan(\theta) & \frac{1}{\varepsilon^2} a \end{pmatrix} \ , \tag{5.5}$$

where $a = a(\varepsilon,\theta) = (1 + (1-\varepsilon^2)\tan^2(\theta))$. Calculating the characteristic form on the hyperboloids $S_\lambda(w)$, we compute the normal $N_\lambda(w)$ as

$$-\frac{1}{2} N_\lambda(w) = (x, Q(z-e_1))^T \ .$$

Noting that the characteristic form is invariant under rotations R as above, which leave the coordinate subspaces $\mathbb{R}_x^{d_1}$ and $\mathbb{R}_y^{d_2}$ invariant, we find that

$$\frac{1}{4} N_\lambda(w)^T \begin{pmatrix} -I_{d_1\times d_1} & 0 \\ 0 & I_{d_2\times d_2} \end{pmatrix} N_\lambda(w) = -|x|^2 + \langle (z-e_1), Q^2(z-e_1)\rangle \ .$$

This is evaluated on the hyperboloid $S_\lambda(w)$, on which

$$|x|^2 + \langle (z-e_1), Q(z-e_1)\rangle = \lambda \ .$$

Solving for $|x|^2$, we find

$$\frac{1}{4}N_\lambda(w)^T \begin{pmatrix} -I_{d_1\times d_1} & 0 \\ 0 & I_{d_2\times d_2} \end{pmatrix} N_\lambda(w) = \langle (z-e_1), [Q^2+Q](z-e_1)\rangle - \lambda \ .$$

Specifically, the matrix $[Q^2+Q]$ is

$$[Q^2+Q] = \begin{pmatrix} Q_2^2+Q_2 & 0 \\ 0 & \left(\frac{1+\varepsilon^2}{\varepsilon^4}\right)I'' \end{pmatrix} \ .$$

Using the form (5.5) for Q_2, one calculates

$$[Q_2^2+Q_2] = \begin{pmatrix} \tan^2(\theta) & \frac{a}{\varepsilon^2}\tan(\theta) \\ \frac{a}{\varepsilon^2}\tan(\theta) & \frac{a^2}{\varepsilon^4}+\tan^2(\theta) \end{pmatrix} \ .$$

It is easily verified that this is positive definite. Recalling that $\lambda \leq 0$ in the definition of the analytic families of hyperboloids, it follows that $S_\lambda(w)$ are all noncharacteristic, and hence the Holmgren–John theorem applies, thus completing the argument. □

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